Multiscaling in Models of Magnetohydrodynamic Turbulence

Abhik Basu,¹ Anirban Sain,¹ Sujan K. Dhar,² and Rahul Pandit¹,*
¹Department of Physics, Indian Institute of Science, Bangalore 560012, India
²Supercomputer Education and Research Center, Indian Institute of Science, Bangalore 560012, India

From a numerical study of the magnetohydrodynamic (MHD) equations we show, for the first time in three dimensions ($d = 3$), that velocity and magnetic-field structure functions exhibit multiscaling, extended self-similarity (ESS), and generalized extended self-similarity (GESS). We propose a new shell model for homogeneous and isotropic MHD turbulence, which preserves all the invariants of ideal MHD, reduces to a well-known shell model for fluid turbulence for zero magnetic field, has no adjustable parameters apart from Reynolds numbers, and exhibits the same multiscaling, ESS, and GESS as the MHD equations. We also study the inertial- to dissipation-range crossover. [S0031-9007(98)07096-3]

The extension of Kolmogorov’s work (K41) [1] on fluid turbulence to magnetohydrodynamic (MHD) turbulence yields [2] simple scaling for velocity $v$ and magnetic-field $b$ structure functions, for distances $r$ in the inertial range between the forcing scale $L$ and the dissipation scale $\eta_d$. Many studies have shown that there are multiscaling corrections to K41 in fluid turbulence [3]. Solar-wind data [4], numerical studies of two-dimensional MHD [5], and recent shell-model studies [6,7] of MHD turbulence yield similar multiscaling. We elucidate this for homogeneous, isotropic MHD turbulence, in the absence of a mean magnetic field, by presenting the first evidence for such multiscaling in a numerical, pseudospectral study of the MHD equations in three dimensions (3dMHD). We propose a shell model with no tunable parameters except Reynolds numbers, study it by an Adams-Bashforth method, show it has this multiscaling, and it reduces to the Gledzer-Ohkitani-Yamada (GOY) shell model [8,9] for 3d fluid turbulence if $b = 0$. To extract multiscaling exponents we develop the ideas of extended self-similarity (ESS) [10,11] and generalized extended self-similarity (GESS) [11,12] in both real and wave-vector ($k$) spaces, used in fluid turbulence [10–12].

We use the structure functions $S_p^a = \langle |a(x + r) − a(x)|^p \rangle$, where $a$ can be $v$, $b$, or one of the ELSs variables $Z^a = v ± b$, $x$ and $r$ are spatial coordinates, and the angular brackets denote an average in the statistical steady state. $S_p^a \sim r^{\xi_p^a}$ at high fluid and magnetic Reynolds numbers $Re$ and $Re_b$, respectively, and for the inertial range $20\eta_d \leq r \ll L$. The extension [2] of K41 to homogeneous, isotropic MHD turbulence with no mean magnetic field yields $\xi_p^a = p/3$. Shell models [6,7] and solar-wind data [4] have obtained multiscaling in MHD turbulence, i.e., $\xi_p^a = p/3 - \xi_p^a$, with $\xi_p^a > 0$ and $\xi_p^a$ nonlinear, monotonically increasing functions of $p$. Work on fluid turbulence shows [3] an extended inertial range if we use ESS [10] and GESS [12]: Thus with ESS, in which $\xi_p^a/\xi_p^3$ follows from $S_p^a \sim [S_3^a]^{\xi_p^a/\xi_p^3}$, we should expect by analogy that it extends down to $r = 5\eta_d$ (as exploited in some MHD shell models [6,7]). In GESS, which employs $G_p^a(r) = S_p^a(r)/[S_3^a(r)]^{p/3}$ and postulates a form $G_p^a(r) \sim (G_3^a(r))^{p/m}$, with $p/m = [\xi_p^a − p \xi_p^3/3]/[\xi_p^a − q \xi_p^3/3]$, it has been suggested [12] for fluid turbulence that the apparent inertial range is extended to the lowest resolvable $r$; however, $k$-space GESS [11] shows a crossover from inertial- to dissipation-range asymptotic behaviors.

Our studies yield many interesting results: The multiscaling exponents we obtain from 3DMHD and our shell model agree (Figs. 1a and 1b) and $\xi_p^a > \xi_3^a \approx \xi_3^b > \xi_3^v$, and $\xi_p^a$ lie close to the She-Leveque (SL) prediction [13] for fluids ($\xi_p^{SL} = p/9 + 2(1 − (2/3)p^{3/2})$, but $\xi_p^a$ lie below it (Fig. 1c) [14]. The probability distribution functions (Fig. 1d) for $\delta v_a(r) = v_a(x + r) − v_a(x)$ and $\delta b_a(r) = b_a(x + r) − b_a(x)$ are also different. ESS works both with real- and $k$-space structure functions (Fig. 2). To study the latter we postulate $k$-space ESS [for real-space structure functions we use $\mathcal{S}$ and $\mathcal{G}$ and for their $k$-space analogs (not Fourier transforms) $\mathcal{S}$ and $\mathcal{G}$]:

$$S_p^a = \langle |a(k)|^p \rangle = A^{a}_{fp}(S_3^a)^{\xi_p^a}/L^{1-\kappa} \leq k \leq 1.5k_d,$$ (1)

$$S_p^a = \langle |a(k)|^p \rangle = A^{a}_{fp}(S_3^a)^{\xi_p^a}, \quad 1.5k_d \leq k \ll \Lambda,$$

where $a(k)$ is the Fourier transform of $a(r)$, $A_{fp}^a$ and $A_{fp}^a$ are, respectively, nonuniversal amplitudes for inertial and dissipation ranges, $k_d \sim \eta_d^{-1}$, and $\Lambda^{-1}$ the (molecular) length at which hydrodynamics breaks down (cf. [11] for fluid turbulence). The exponents $\alpha_p^a$ and $\xi_p^a$ characterize the asymptotic behaviors of the structure functions in dissipation and inertial ranges. They are universal, but $\alpha_p^a \neq \xi_p^a$. In our shell model $\xi_p^a = \xi_3^a$, but our data for 3DMHD suggest $\xi_p^a = 2(\xi_p^v + 3p/2)/11$ (i.e., $S_p^a(k) \sim k^{-(\xi_p^a + 3p/2)}$ in the inertial range [15]); the difference arises because of phase-space factors [11]. $\xi_p^a$ and $\alpha_p^a$ seem universal (the same for all our runs [Table I]); $\alpha_p^a$ is close to, but systematically less than, $p/3$. The $k$ dependences
of $S_t^a$ follow from that of $S_3^a$. We find

$$S_t^a = B_t^d k^{\delta^o} \exp(-c^a k/d), \quad 1.5 k_d \leq k \ll \Lambda,$$  

(3)

where $B_t^d$ and $B_D^d$ are nonuniversal amplitudes [Equation (2) holds [11] for 3DMHD: for our shell model the factor 9/2 is absent]. Thus all $S_t^a - k^{\theta^o} \exp(-c^a k/d) \log(\log(k/d))$ for $1.5 k_d \leq k \ll \Lambda$, with $\theta^o = \alpha^o \delta^o$ (cf. [11] for fluid turbulence). In Eq. (3) $\delta^o$, $c^o$, and $k_d$ are not universal; they depend on whether we use the 3DMHD or our shell model. We extract the universal part of the inertial- to dissipation-range crossover via our $k$-space GESS as follows: We first define $G_p^a \equiv S_p^a/(S_3^a)^{p/3}$; log-log plots of $G_p^a$ versus $G_q^a$ yield curves with universal, but different, slopes for asymptotes in inertial and dissipation ranges. The inertial-range asymptote has a slope $\rho_{nu}^{a}(p)$ (as in real-space GESS); the dissipation-range one has a slope $\omega(p, q) \equiv [\alpha^a - p/3]/[\alpha^a - q/3]$. These slopes are universal, but not the points at which the curves move away from the inertial-range asymptote.

To obtain a universal crossover scaling function [different for each $(p, q)$ pair because of multiscaling] we define $\log(H_p^a) = D_p^a \log(G_p^a)$ and $\log(H_q^a) = D_q^a \log(G_q^a)$; the scale factors $D_p^a = D_q^a$ are nonuniversal, but plots of $\log(H_p^a)$ versus $\log(H_q^a)$, for both 3DMHD and our shell model, collapse onto a universal curve within our error bars for all $k$, Re$_\Lambda$, and Re$_{h, k}$ (Fig. 3).

The MHD equations are [2]

$$\frac{\partial \mathbf{Z}}{\partial t} + (\mathbf{Z} \cdot \nabla) \mathbf{Z} = \nu_+ \nabla^2 \mathbf{Z} + \nu_- \nabla^2 \mathbf{Z},$$

(4)

where $\nu_\pm = (\nu_+ \pm \nu_\pm)/2$, $\nu_+ \nu_\pm$ and $\nu_\pm$ are, respectively, fluid and magnetic viscosities, $p^a = [p + (b^2/8\pi)]$.

### Table 1. The viscosities and hyperviscosities $\nu_v$, $\nu_h$, $\nu_{vH}$, and $\nu_{vhl}$, the Taylor-microscale Reynolds numbers Re$_{a}$ and $Re_{h}$, the box-size eddy-turnover times $\tau_\nu$ and $\tau_{ed}$, the averaging time $\tau_a$, the time over which transients are allowed to decay $\tau_r$, and $k_d$ (dissipation-scale wave number) for our 3DMHD runs ($k_{max} = 32$ for MHD1 and MHD2 and $k_{max} = 40$ for MHD3) and shell-model runs SH1–4 ($k_{max} = 2^{5}k_0$). The step size $\Delta t$ is 0.02 for MHD1–3, $2 \times 10^{-5}$ for SH1–2, and $4 \times 10^{-4}$ for SH3–4. Note that $\tau_{ed} = 8\tau_a$ the integral time for our 3DMHD runs.

<table>
<thead>
<tr>
<th>Run</th>
<th>$\nu_v$</th>
<th>$\nu_{vH}$</th>
<th>$\nu_h$</th>
<th>$\nu_{vhl}$</th>
<th>$Re_a$</th>
<th>$Re_{h}$</th>
<th>$\tau_\nu/\Delta t$</th>
<th>$\tau_{ed}/\Delta t$</th>
<th>$\tau_r/\tau_\nu$</th>
<th>$\tau_a/\tau_\nu$</th>
<th>$k_{max}/k_d$</th>
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<tbody>
<tr>
<td>MHD1</td>
<td>$8 \times 10^{-4}$</td>
<td>$7 \times 10^{-6}$</td>
<td>$10^{-3}$</td>
<td>$8 \times 10^{-6}$</td>
<td>$\approx 24.8$</td>
<td>$\approx 14.3$</td>
<td>$\approx 8.8 \times 10^3$</td>
<td>$\approx 6 \times 10^3$</td>
<td>$\approx 2$</td>
<td>$\approx 2.3$</td>
<td>$\approx 1.83$</td>
</tr>
<tr>
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<td>$8 \times 10^{-4}$</td>
<td>$9 \times 10^{-6}$</td>
<td>$8 \times 10^{-4}$</td>
<td>$9 \times 10^{-6}$</td>
<td>$\approx 24.1$</td>
<td>$\approx 18.1$</td>
<td>$\approx 8.8 \times 10^3$</td>
<td>$\approx 5.6 \times 10^3$</td>
<td>$\approx 2$</td>
<td>$\approx 2.3$</td>
<td>$\approx 1.83$</td>
</tr>
<tr>
<td>MHD3</td>
<td>$8 \times 10^{-4}$</td>
<td>$9 \times 10^{-6}$</td>
<td>$9 \times 10^{-4}$</td>
<td>$9 \times 10^{-6}$</td>
<td>$\approx 26$</td>
<td>$\approx 19.6$</td>
<td>$\approx 7.9 \times 10^3$</td>
<td>$\approx 4.8 \times 10^3$</td>
<td>$\approx 1$</td>
<td>$\approx 2.2$</td>
<td>$\approx 2.22$</td>
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<tr>
<td>SH1</td>
<td>$10^{-9}$</td>
<td>$0$</td>
<td>$10^{-8}$</td>
<td>$0$</td>
<td>$\approx 4.6 \times 10^8$</td>
<td>$\approx 7.8 \times 10^8$</td>
<td>$\approx 10^7$</td>
<td>$\approx 6 \times 10^6$</td>
<td>$\approx 50$</td>
<td>$\approx 450$</td>
<td>$\approx 25^8$</td>
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<tr>
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<td>$0$</td>
<td>$10^{-8}$</td>
<td>$0$</td>
<td>$\approx 4.3 \times 10^7$</td>
<td>$\approx 6.5 \times 10^7$</td>
<td>$\approx 10^6$</td>
<td>$\approx 6 \times 10^6$</td>
<td>$\approx 50$</td>
<td>$\approx 450$</td>
<td>$\approx 28^8$</td>
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<tr>
<td>SH3</td>
<td>$10^{-6}$</td>
<td>$0$</td>
<td>$2 \times 10^{-6}$</td>
<td>$0$</td>
<td>$\approx 4 \times 10^6$</td>
<td>$\approx 3 \times 10^6$</td>
<td>$\approx 2 \times 10^6$</td>
<td>$\approx 10^5$</td>
<td>$\approx 500$</td>
<td>$\approx 2500$</td>
<td>$\approx 2^10$</td>
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<td>$0$</td>
<td>$10^{-6}$</td>
<td>$0$</td>
<td>$\approx 1.2 \times 10^5$</td>
<td>$\approx 1 \times 10^5$</td>
<td>$\approx 10^5$</td>
<td>$\approx 1.7 \times 10^5$</td>
<td>$\approx 500$</td>
<td>$\approx 3000$</td>
<td>$\approx 2^{11}$</td>
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with $p$ the pressure, the density $\rho = 1$, $f^\pm = (f \pm g)/2$, and $f$ and $g$ are the forcing terms in the equations for $\partial v/\partial t$ and $\partial b/\partial t$. We assume incompressibility and use a pseudospectral method \cite{11} to solve Eq. (4) numerically. We force the first two $k$ shells, use a cubical box with side $L_B = 2\pi$, periodic boundary conditions, and $64^3$ modes in runs MHD1 and MHD2 and $80^3$ modes in run MHD3 (Table I). We include fluid and magnetic hyperviscosities $\nu_{vH}$ and $\nu_{BH}$ [i.e., the term $-(\nu_v + \nu_{vH}k^2)b^2$ in the equation for $\partial v(k)/\partial t$ and the term $-(\nu_B + \nu_{BH}k^2)b^2$ in the equation for $\partial b(k)/\partial t$] \cite{16}. For time integration we use an Adams-Bashforth scheme (step-size $\delta t$). We use $Re_\lambda = \nu_{vH}\lambda/\nu_v$, $Re_{BH} = \nu_{BH}\lambda/b$.

We used for inertial-range fitting).

Points on the right correspond to forcing scales and are not model; the lines show the inertial-range asymptotes (a few illustrate real-space ESS for 3DMHD and ESS for our shell model and show that it yields $\zeta^{\mu}_n$ in agreement with those we obtain for 3DMHD. Our shell-model equations

$$\frac{dz_n^\pm}{dt} = ic_n^\mp - \nu k_n^2z_n^\mp - \nu k_n^2z_n^\mp + f_n^\pm$$

(5)

use the complex, scalar Elsässer variables $z_n^\pm = (\nu_n \pm b_n)$, and discrete wave vectors $k_n = k_nq^n$, for shells $n$; $c_n = [a_1k_n^2\zeta_n^2 + a_2k_n^2\zeta_{n+2}^2 + a_3k_n^2\zeta_{n-2}^2 + a_4k_n^{-1}\zeta_n^2 + a_5k_n^{-1}\zeta_{n+2}^2 + a_6k_n^{-1}\zeta_{n-2}^2]$, which ensures $z_n^\pm$, $\zeta_n^\pm \sim k^{-1/3}$ is a stationary solution in the inviscid, unforced limit.
[6–9] and preserves the $\nu_+ Z^+ \leftrightarrow \nu_- Z^-$ symmetry of 3DMHD. We fix five of the parameters, $a_1 - a_6$, by demanding that our shell-model analogs of the total energy $\left[= \sum_n \left(\nu_n v_n^2 + |b_n|^2\right)/2\right]$, the cross helicity $\left[= 1/2 \sum_n \left(\nu_n b_n^* v_n b_n + v_n^* b_n^* b_n\right)\right]$, and the magnetic helicity $\left[= \sum_n (a_n)^{-1} |b_n|^2 k_n\right]$ be conserved if $\nu_\pm = 0$ and $f_n^\pm = 0$; while enforcing the conservation of energy, we also demand [18] that the cancellation of terms occurs as in 3DMHD. We fix the last parameter by demanding that, if $b_n = 0$ for all $n$, our model reduces to the GOY model, with the standard parameters [9] that enforce conservation laws. Finally $a_1 = 7/12$, $a_2 = 5/12$, $a_3 = -1/12$, $a_4 = -5/12$, $a_5 = -7/12$, $a_6 = 1/12$, and $q = 2$. We solve Eq. (5) numerically by an Adams-Bashforth scheme (step size $\delta t$), use 25 shells, force the first $k$ shell [11], set $k_0 = 2^{-4} = 1/(2L_3)$, where $L_3$ is the box size, and use $E_\nu = S_3^{\nu} k_k^2 / k_n$, $\lambda_\nu = (2\pi/k_0) \left[\sum_n S_n^{\nu} (k_n)/\sum_n k_n^2 S_2^{\nu} (k_n)\right]^{1/2}$, $\lambda_b = (2\pi/k_0) \left[\sum_n S_n^{\nu} S_2^{\nu} (k_n)/\sum_n k_n^2 S_2^{\nu} (k_n)\right]^{1/2}$, $u_{rms} = \left[\sum_n k_n^2 S_2^{\nu} (k_n)/\pi\right]^{1/2}$, and $b_{rms} = \left[\sum_n k_n^2 S_2^{\nu} (k_n)/\pi\right]^{1/2}$. Parameters for our four runs SH1–SH4 are given in Table I. These use double-precision arithmetic, but we have checked in representative cases that our results are not affected if we use quadruple-precision arithmetic. As in the GOY model the structure functions $S_\nu^{\nu}(k_n)$ oscillate weakly with $k_n$ because of an underlying three cycle [9,18]. These oscillations can be removed either (a) by using ESS or (b) by using the structure functions $S_{n,p} = \left(\alpha_{n+1} a_{n+2} + a_n - \alpha_{n-1} a_{n-2}\right) / 4 \right)^{1/3}$ [9]. Method (a) yields $\xi_p^b / \xi_p^g$, which we find are universal. Method (b) gives exponents $\xi_p^g$. These have a mild dependence on Re$_b$ and Re$_{b,h}$ but this goes away if we consider the ratios $\xi_p^b / \xi_p^g$, as in the GOY model [11]; thus the asymptotes in our ESS and GESS plots have universal slopes.

The Navier Stokes equation (3DNS) follows from 3DMHD if $b = 0$ or, equivalently, Re$_{b,h} = 0$. However, if we start with Re$_{b,h} = 0$, the steady state is characterized by the MHD exponents and Re$_b$ = Re$_{b,h} = 0$ (1), i.e., an equipartition regime [19]. Since our MHD shell model reduces to the GOY model as Re$_b$ → 0, we use it (and not costly 3DMHD) to study the fluid turbulence to MHD turbulence crossover: A small initial value of Re$_{b,h}$ yields a transient with GOY-model exponents, but finally the system crosses over to the MHD turbulence steady state [18].

In summary, we have shown that structure functions in 3DMHD turbulence display multiscaling, ESS, and GESS, with exponents and probability distributions different from those in fluid turbulence. Our new shell model (a) gives the same exponents as 3DMHD and (b) reduces to the GOY model as Re$_{b,h} \to 0$. Our ESS and GESS uncover a universal crossover from inertial- to dissipation-range asymptotics. It would be interesting to compare our results with experiments, but with caution: (i) solar-wind data might yield exponents different from ours because of the presence of a mean magnetic field and compressive effects [20]; (ii) the inertial- to dissipation-range crossover might not apply to the solar wind because a hydrodynamic description might break down in the dissipation range [20]. However, our results should apply to MHD systems with an equipartition regime [2]. The agreement of $\xi_p^b$ with the SL formula is interesting but, we believe, fortuitous since vorticity organizes itself into filamentary structures [13] in fluid turbulence but into sheetlike structures in 3DMHD (we have checked this in our study).

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*Also at Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore, India.

[14] We use the SL formula as a convenient parametrization for the multiscaling exponents in fluid turbulence.
[19] Thus in a renormalization-group calculation Re$_{b,h}$ should appear as a relevant operator.