it follows that
\[ \int_0^{t_m} |w_i(t)| \, dt \leq \left( \int_0^{t_m} \left| \sum_{i=1}^n \delta_{i,j} \frac{\partial f_i}{\partial x_j}(x,v) \right| \, dt \right)^{1/2} \]
(24)
(25)
Recalling (22), if \( w_i(t) = 0 \) for \( t > t_m \), the inequality (24) can be written in the form
\[ \int_0^{t_m} |w_i(t)| \, dt \leq \left( \sum_j |W_{i,j}(w_i)|^2 \right)^{1/2} \]
The evaluation of the term in the right-hand side of (25) is very easy if \( W_k(x) \) is of the form
\[ W_k(x) = \sum_{i=1}^n a_i^k x_i + \sum_{i=1}^n b_i^k x_i^2 \]
with \( n > m \), \( \sum_i b_i^k \) strictly Hurwitzian, and all the poles \( p_i \) simple. Then,
\[ \left( \sum_j |W_{i,j}(w_i)|^2 \right)^{1/2} \]

The accuracy of the approximation of the integral (17) by the right-hand side of (25) depends on \( t_m \) since the \( L_2 \) norm approaches the \( L_1 \) norm as \( t_m \) goes to zero.

Conclusions
A sufficient condition is derived here for the boundedness of the responses of a class of nonlinear feedback systems to amplitude limited signals. This condition depends on the linear part of the system and on the range of the derivative of the nonlinear characteristic and on an upper bound on the responses is then determined. On the other hand, it is possible to find an interval, if it exists, in which the amplitude of the responses is within a bound fixed a priori.

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References

Stability of a System of Linear Differential Equations

Abstract—Conditions for asymptotic stability of a system of linear time-invariant differential equations of a particular type with matrix coefficients are shown to be similar to those obtained through the Routh–Hurwitz inequalities, but wherein real numbers and positivity are replaced by the matrix coefficients and positive definiteness.

Introduction
Seshu and Reed have mentioned the conditions under which a second-order linear time-invariant differential equation with real constant matrix coefficients is stable.1 Ezzelbo established stability conditions for a system of \( n \) third-order nonlinear differential equations of a particular class,2

\[ \sum_{i=1}^{n} \left[ a_i \frac{d^2 x_i}{dt^2} + b_i \frac{dx_i}{dt} + c_i x_i \right] = 0, \quad i = 1, \ldots, n \]

It is impossible to find a relation similar to (25) if the poles of \( W_k(x) \) are not simple. The accuracy of the approximation of the integral (17) by the right-hand side of (25) depends on \( t_m \) since the \( L_2 \) norm approaches the \( L_1 \) norm as \( t_m \) goes to zero.


\[ \sum_{i=1}^{n} \left[ a_i \frac{d^2 x_i}{dt^2} + b_i \frac{dx_i}{dt} + c_i x_i \right] = 0, \quad i = 1, \ldots, n \]

In this correspondence, conditions for asymptotic stability in the large of \( m \)-order ordinary differential equations with time-invariant matrix coefficients are derived.

Notation
Let \( R^m \) and \( T_r \) denote the real Euclidean \( m \)-dimensional space and the real interval \( 0 \leq t < \infty \), respectively. Consider a system of \( m \) \( n \)-th order differential equations

\[ x^{(n)} + B_1 X^{(n-1)} + B_2 X^{(n-2)} + \cdots + B_{n-1} X^2 + B_n X = 0 \]

where \( X \in R^m \), \( X \) is an \( m \) vector, and \( B_i, i = 1, 2, \ldots, n \), are \( m \times m \) real constant matrices. It is assumed that (1) satisfies all the requirements for uniqueness and existence of solution \( \phi(t; x_1, \ldots, x_n; t_0) \) which corresponds to the initial state \( (x_1, x_2, \ldots, x_n) \) in the motion space \( R^m \times T_r \), where the vectors \( x_1, x_2, \ldots, x_n \) represent the successive time derivatives of the vector \( X(t) = x_1 \). Equation (1) is a vector version of the system of real linear time-invariant \( m \)-th order differential equations

\[ x^{(n)} + \sum_{i=1}^{n} b_{i1} x_i + \cdots + \sum_{i=1}^{n} b_{in} x_i = 0, \quad j = 1, \ldots, n \]

Definition 1: The system of differential equations (1) is said to be asymptotically stable if the vectors \( X(t) \) converge to zero as time \( t \to \infty \). The vectors \( X(t) \to 0 \) imply that every component of \( X(t) \) the \( x_i(t) \) \( (j = 1, \ldots, n) \to 0 \) as \( t \to \infty \).

Definition 2: The determinant formed, similar to that of the Routh–Hurwitz determinant but with matrix coefficients \( [B_i] \) of (1) treated in place of real numbers, will be called the matrix Routh–Hurwitz determinant.

Theorem 1
If the matrix coefficients \( [B_i] \) of (1) can be simultaneously diagonalized by a non-singular transformation, then positive definiteness of all minors \( \Delta_i (i = 1, \ldots, n) \) of the matrix Routh–Hurwitz determinant are necessary and sufficient criteria for asymptotic stability of the system of equations (1).

Proof: Let the matrix \( [L] \) diagonalize the matrices \( [B_i] \) such that matrices \( [A_i] \) are diagonal with the transformation,

\[ [L]^{-1} [B_i] [L] = [A_i], \quad i = 1, \ldots, n \]

Then the positive definiteness of the product of matrices

\[ [B_i] [B_j], \quad i = 1, \ldots, n \]

is equivalent to the positive definiteness of the product of matrices

\[ [A_i] [A_j], \quad i = 1, \ldots, n \]

since

\[ [A_i] \cdot [B_j] = [L]^{-1} [B_i] [L] \cdot [L]^{-1} [B_j] [L] = [L]^{-1} [B_i] [L] \cdot [B_j] [L] \]
In this new set of coordinates, the uncontrollable (unobservable) state variables are easily identified and removed. An example is given.

It is well known\(^1\) that a system which is uncontrollable and/or unobservable can be reduced to a zero-state equivalent system of lower dynamic order by removing the uncontrollable and unobservable modes. This problem has been considered by Silverman and Meadows,\(^{11}\) Glass and D’Angelo,\(^{14}\) and others. It is the purpose of this correspondence to present a very simple procedure for removing uncontrollable and unobservable state variables of time-variant linear systems to yield a zero-state equivalent system with the minimum number of state variables. In earlier papers,\(^{10},^{11},^{12}\) algorithms for constructing minimum-state realizations of time-invariant systems were presented. Some of these techniques will be applied to the time-varying system problem.

The system is represented by the state equations

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) \\
\dot{y}(t) = C(t)x(t)
\]

where \(A(t)\), \(B(t)\), and \(C(t)\) are \(n \times n\), \(n \times m\), and \(p \times n\) matrices, respectively, with possibly time-varying elements, and where matrices \(A(t)\), \(B(t)\), and \(C(t)\) together with their first \(n-2\), \(n-1\), and \(n-2\) derivatives, respectively, are continuous functions, then any uncontrollable state variables in the system may be removed as follows.

1. Form the controllability matrix \(Q_{c}(t)\) as given in (2).
2. Augment \(Q_{c}(t)\) with an \(n \times n\) identity matrix, i.e.,

\[
[Q_{c}(t); I].
\]
3. Transform \(Q_{c}(t)\) into Hermite normal (row-echelon) form \((Q_{c}(t))\) using elementary row operations and obtain the transformation matrix \(T_{c}\) which must be nonsingular and differentiable, where

\[
[Q_{c}(t); I] \Rightarrow [Q_{c}(t); T_{c}^{-1}].
\]
4. Identify the uncontrollable state variables of the system, in this new set of coordinates, as those state variables corresponding to the all-zero rows of \(Q_{c}(t)\) where row \(i\) corresponds to state variable \(x_{i}\). Transform the system \(\hat{S} = (A, B, C)\) into the new coordinate system \(\hat{S} = (A, B, C)\) where

\[
A = (T_{c}^{-1}AT_{c} - T_{c}^{-1}T_{c}B) \\
B = T_{c}^{-1}B \\
C = CT_{c}.
\]

5. For each uncontrollable state variable \(x_{i}\) as identified in step 4, delete row \(i\) and column \(i\) from \(A\), row \(i\) from \(B\), and column \(i\) from \(C\) to obtain the reduced system

\[
\hat{S} = (A, B, C).
\]

Proof of this algorithm follows directly from Theorem 5.1 of Nering,\(^{2}\) Lemma 1 of Silverman and Meadows,\(^{11}\) and Corollary 11.3.15 of Zadah and Desoer,\(^{27}\) and is presented in Albertson,\(^{28}\) but it will not be given here because of space limitations.

The algorithm for removing unobservable modes is the dual of Algorithm 1, where \(Q_{c}(t)\) given in (3) is utilized and where the transformation

\[
[Q_{c}(t); I] \Rightarrow [Q_{c}(t); T_{c}].
\]

is carried out using elementary row operations. The matrix \(T_{c}\) is the transpose of the desired transformation matrix.

Example

The following example, considered previously,\(^{10},^{11}\) illustrates the use of the algorithm. Consider system (1) where

\[
A(t) = \begin{bmatrix}
-t -1 & 0 & -t + 2 \\
1 & t & 1 + t \\
0 & -t + 1
\end{bmatrix}
\]

\[
B(t) = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

\[
C(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The controllability matrix using (2) is

\[
Q_{c}(t) = \begin{bmatrix}
-1 & -t & 0 \\
1 & 1 + t & 2 \\
0 & -t & -1
\end{bmatrix}
\]

Now, using (4),

\[
Q_{c}(t) = \begin{bmatrix}
1 & 1 - t & 0 \\
1 & 1 + t & 2 \\
0 & -t - 1 & 0
\end{bmatrix}
\]

It is noted that \(Q_{c}(t)\) has rank 2 everywhere and that state variable \(x_{1}\) in the new coordinate system, is uncontrollable. Also note that

\[
T_{c}^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 1 \\
0 & -1 & 0
\end{bmatrix}
\]

\(^{1}\) It may not always be necessary to reduce \(Q_{c}(t)\) completely to Hermite form, but only to the point where the all-zero rows of \(Q_{c}(t)\) are identified (see example).

\(^{2}\) See page 689 of Silverman and Meadows\(^{11}\) for the definition of total controllability.