Forming Networks of Strategic Agents with Desired Topologies

Swapnil Dhamal and Y. Narahari
Indian Institute of Science, Bangalore, India

Abstract

Many networks such as social networks and organizational networks in global companies consist of self-interested agents. The topology of these networks often plays a crucial role in important tasks such as information diffusion and information extraction. Consequently, growing a stable network having a certain topology is of interest. Motivated by this, we study the following important problem: given a certain desired network topology, under what conditions would best response (link addition/deletion) strategies played by self-interested agents lead to formation of a stable network having that topology. We study this interesting reverse engineering problem by proposing a natural model of recursive network formation and a utility model that captures many key features. Based on this model, we analyze relevant network topologies and derive a set of sufficient conditions under which these topologies emerge as pairwise stable networks, wherein no node wants to delete any of its links and no two nodes would want to create a link between them.

Keywords: Social Networks, Network Formation, Pairwise Stability, Network Topology, Strategic Agents.

1 Introduction

In a social network, individuals gain certain benefits from other individuals and at the same time, pay a certain cost for maintaining links with their friends. Owing to the tension between benefits and costs, self-interested or rational nodes think strategically while choosing their immediate neighbors. A stable network that forms out of this process will have a topological structure as dictated by the individual utilities and best response strategies of the nodes.

Often, stakeholders such as a social network owner or a social planner, who work with the networks so formed, would like the network to have a certain desirable topology to accomplish certain tasks. Typical examples of these tasks include enabling optimal communication among nodes for maximum efficiency (knowledge management), extracting certain critical information from the nodes (information extraction), broadcasting some information to the nodes (information diffusion), etc. If a particular topology is the most appropriate for the set of tasks to be handled, it would be useful to orchestrate network formation in a way that the required topology emerges as a stable configuration as a culmination of the network formation process.

1.1 Motivation

One of the key problems addressed in the literature on social network formation is: given a set of self-interested nodes and a model of social network formation, which topologies would be stable and which would be efficient (maximizing sum of utilities of all nodes). In this paper, our focus is on the inverse problem, namely, given a certain desired topology, under what conditions would best response strategies played by self-interested agents lead to formation of a stable network with that topology. We motivate this problem with some relevant topologies.

Consider a network where there is a need to rapidly spread some crucial information received by any of the nodes, requiring precautions against link failures. In such cases, a complete network is ideal. Consider a different scenario where the information needs to be spread rapidly, however there needs to be a moderator to verify the authenticity of the information before spreading it to the other nodes in the network (for example, it could be a rumor). Here a star network is desirable. Consider a generalization of the star network where there is a need for decentralization for efficiently controlling information in the network. It has multiple centers, each linked to every other, and the leaf nodes are divided among the centers as evenly as possible. We call it, k-star network. Consider a necessity of having two sections where some or all members of a section receive certain information simultaneously and there is a need to forward it to the other section, taking care of link failures. Moreover, it is desirable to not have intra-section links to save on resources. A bipartite Turán network is ideal in this case as both communities are practically desirable to be of nearly equal size.

It is clear that depending on the tasks for which the network is used, a certain topology might be better than others. This provides the motivation for our work.

1.2 Relevant Work

Jackson [5] reviews several models of network formation in the literature. Watts [10] provides a sequential move game model where nodes are myopic; however, the resulting network is based on the ordering in which links are altered and so it is unclear which networks emerge [6]. Hummon [4] uses simulations to explore the dynamics of network evolution. Doreian [2] analytically arrives at specific networks that are pairwise stable; but its complexity increases exponentially with the number of nodes and so the analysis is limited to only five nodes.

There have been a few approaches to design incentives for nodes so that the resulting network is efficient [8, 11]. Though it is often assumed that the welfare of a network is based

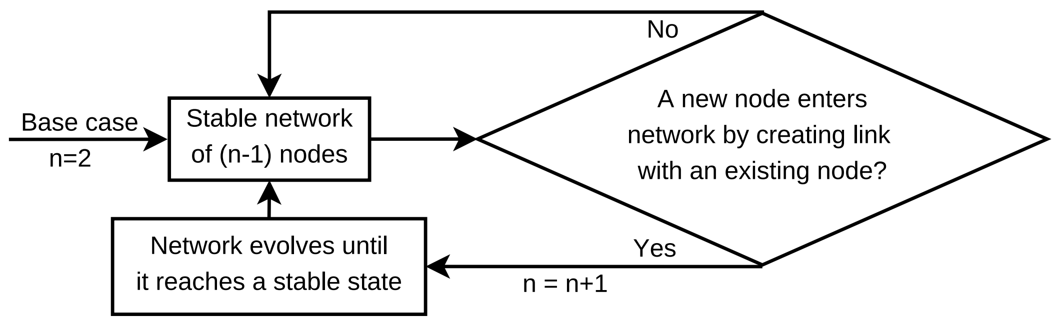

Figure 1: Proposed model of network formation

only on its efficiency, there are many situations where this may not be true. A particular network may not be efficient in itself, but it may be desirable for reasons external to the network, as explained in Section 1.1.

The models of social network formation in literature assume that all nodes are present throughout the evolution of a network, which allows nodes to form links that may not be consistent with the desired network. Furthermore, with all nodes present in an unorganized network, a random ordering over them in sequential network formation models adds to the complexity of analysis. However, in most social networks, not all nodes are present from beginning itself. A network starts building up from a few nodes and gradually grows to its capacity.

1.3 Contributions of the Paper

- We propose a recursive model of network formation using which it is possible to guarantee that the network retains its topology in each of its stable states; also the analysis can be carried out independent of the current number of nodes in the network. We also propose a utility model that captures many key features, including an *entry fee* for entering the network.

- We derive sufficient conditions under which star network, complete network, bipartite Turán network, and k-star network, emerge as pairwise stable.

2 A Recursive Model of Network Formation

The game is played amongst self-interested nodes, which we consider to be all homogeneous and having global knowledge of the network[1]. Each node, which gets to make a move, has a set of strategies at any given time and it chooses its myopic best response strategy[2]. A

[1] As assumed in most of the literature on social network formation [6]

[2] The assumption of nodes behaving myopically has experimental justifications [9].

strategy can be of one of the three types, namely (a) creating a link with a node that is not its immediate neighbor with its consent, (b) deleting a link with an immediate neighbor without its consent, or (c) maintaining status quo. Moreover, consistent with the notion of pairwise stability, if a node gets to make a move, and proposing or deleting a link does not strictly increase its utility, then it prefers not to do so. But a node will accept a link proposed by some other node provided its utility does not decrease.

The game starts with one node and the process goes on as depicted in Figure 1. Now given that a stable network of $n-1$ nodes is formed, the n^{th} node considers entering the network. We make an intuitive assumption that in order to be a part of the network, the n^{th} node has to propose a link with one of the existing nodes and not vice versa. For successful link creation, utility of the latter should not decrease. After the new node enters the network, nodes who get to make their move are chosen at random at all time and the network evolves until it becomes a stable network consisting of n nodes. Following this, a new $(n+1)^{th}$ node considers entering the network and the process goes on recursively[3].

2.1 Utility Model

Our utility model takes the idea of essential nodes proposed by Goyal and Vega-Redondo [3]. A node j is said to be essential for y and z if j lies on every path that joins y and z in the network. Any two nodes pay a fraction of the benefits obtained from each other, as intermediation rents in the form of additional favors or monetary transfers to the corresponding set of essential nodes[4]. This fraction is assumed to be equally divided among the essential nodes connecting that pair. Thus, nodes get bridging benefits for being an essential node for each such pair.

We introduce a notion of network entry fee which corresponds to some cost a node has to bear in order to be a part of the network. If a newly entering node wants its first connection to be with an existing node of high importance or degree, then it has to spend more time or effort. So we assume the entry fee that the former pays to be an increasing function of the degree of the latter, say d_T. For simplicity of analysis, we assume the fee to be directly proportional to d_T and call the proportionality constant, network entry factor c_0.

Table 1 enlists the notation used in the paper. For a node j, the utility function is a function of the network, that is $u_j : g \to \mathbb{R}$ and is given by

[3]The assumption that a node considers entering the network only when it is pairwise stable might seem artificial in general social networks, but can be justified in organizational networks where entry of nodes can be controlled by an administrator.

[4]In order to avoid discrete constraints on rents, such as summation of the fractions paid to be less than one, we assume that irrespective of the number of essential nodes (provided positive) connecting y and z, they lose the same fraction $\gamma \in [0, 1)$. As a result, the real producers of benefits are guaranteed at least $(1-\gamma)$ fraction of it.

N	set of nodes present in the network
u_j	net utility that node j gets from the network
d_j	degree of node j
b_i	benefits obtained from a node at distance i (where $b_{i+1} < b_i$)
c	cost incurred in maintaining link with an immediate neighbor
$l(j,w)$	shortest path distance between nodes j and w
$E(j,w)$	set of nodes essential to connect j and w
γ	fraction of indirect benefits paid to the corresponding set of essential nodes
c_0	network entry factor (see Section 2.1)
$\mathrm{T}(j)$	existing node in the network to which node j connects to enter the network
$\mathbf{I}_{\{j=\mathrm{NE}\}}$	1 when j is the newly entering node about to create its first link, else 0

Table 1: Notation for the proposed utility model

$$u_j = -c_0 d_{\mathrm{T}(j)} \mathbf{I}_{\{j=\mathrm{NE}\}} + d_j b_1 - d_j c + \sum_{\substack{w \in N \\ l(j,w)>1}} b_{l(j,w)} - \sum_{\substack{w \in N \\ E(j,w) \neq \phi}} \gamma b_{l(j,w)} + \sum_{\substack{y,z \in N \\ j \in E(y,z)}} \left(\frac{\gamma}{|E(y,z)|} \right) 2 b_{l(y,z)} \tag{1}$$

The individual terms of Equation (1) represent (a) network entry fee, (b) benefits from immediate neighbors, (c) costs of maintaining links with immediate neighbors, (d) benefits from indirect neighbors, (e) intermediation rents paid, and (f) bridging benefits, respectively.

2.2 Directing Network Evolution

We consider the sequential move game model and so the process of network evolution can be represented as a game tree. The entry of each node in the network results in a game tree. An *improving path* is a sequence of networks, where each transition is obtained by either two nodes choosing to add a link or one node choosing to delete a link [7]. Thus, a pairwise stable network is one from which there is no improving path leaving it. Hence, our objective is to direct the network evolution along a desired improving path in the game tree.

The procedure for deriving sufficient conditions for the formation of a given topology is similar to *mathematical induction*. Consider a base case network with very few nodes (two in our analysis). We derive conditions so that the network formed with these few nodes has the desired topology. Then using induction, we assume that a network with $n-1$ nodes has the desired topology, and derive conditions so that, the network with n nodes, also has that topology.

In Figure 2, we direct the network evolution by imposing a set of conditions ensuring that the resulting pairwise stable network is a star. Let $u_j(s)$ be the utility of node j when the network is in state s and let $leaf \in \{C, D, E, F\}$. As all leaf nodes are equivalent up

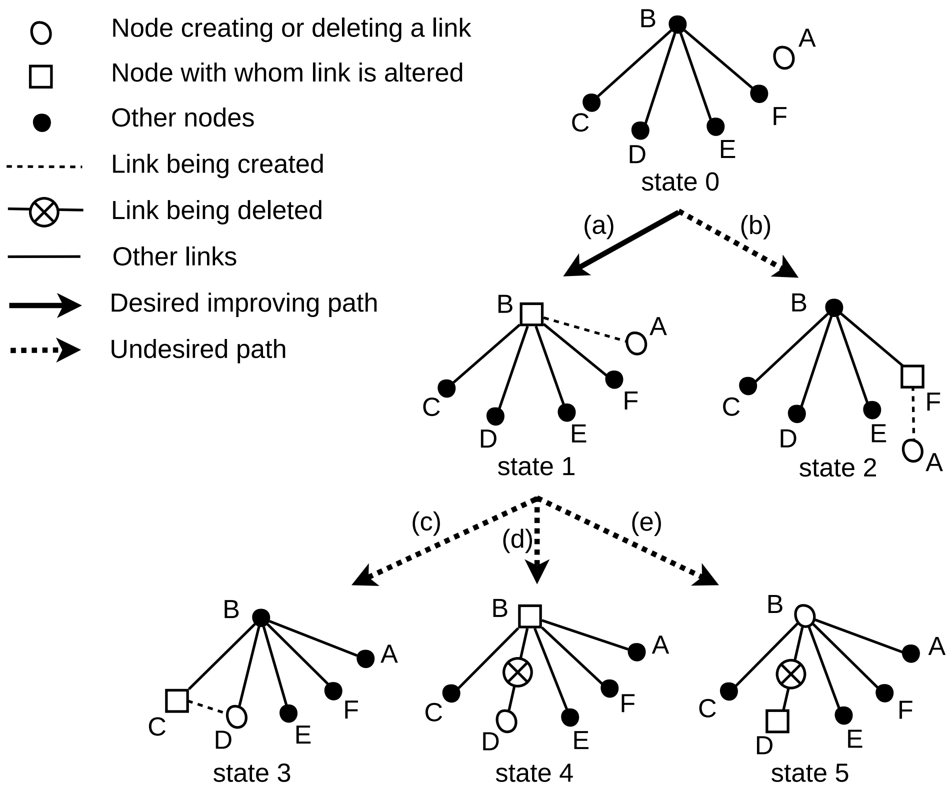

Figure 2: Directing Network Evolution for the Formation of Star Topology

to relabeling, considering utility of one such node is sufficient. The conditions sufficient to direct the network evolution along the desired improving path and avoid any undesired paths (be they improving or not) are (a) $u_A(1) > u_A(0)$ and $u_B(1) \geq u_B(0)$, (b) $u_A(1) > u_A(2)$ or $u_{leaf}(2) < u_{leaf}(0)$, (c) $u_{leaf}(1) \geq u_{leaf}(3)$, (d) $u_{leaf}(1) \geq u_{leaf}(4)$, and (e) $u_B(1) \geq u_B(5)$.

3 Sufficient Conditions for Relevant Topologies

In this section, we provide sufficient conditions for the formation of relevant topologies. We use Equation (1) for mathematically deriving these conditions. For the proofs, the reader is referred to the full version of this paper [1]. It also shows that with the derived sufficient conditions, star network and complete network are efficient, and for sufficiently large number of nodes, efficiencies of bipartite Turán network and k-star network are respectively, half and $\frac{1}{k}$ of that of the efficient network in the worst case and the networks are close to being efficient in the best case.

Theorem 1 *For a network, if $b_1 - b_2 + \gamma b_2 \leq c < b_1$ and $c_0 < (1-\gamma)(b_2 - b_3)$, the resulting topology is a star graph.*

Theorem 2 *For a network, if* $c < b_1 - b_2$ *and* $c_0 \leq (1-\gamma)\, b_2$*, the resulting topology is a complete graph.*

Theorem 3 *For a network with* $\gamma < \frac{b_2 - b_3}{3b_2 - b_3}$*, if* $b_1 - b_2 + \gamma\,(3b_2 - b_3) < c < b_1 - b_3$ *and* $(1-\gamma)\,(b_2 - b_3) < c_0 \leq (1-\gamma)\, b_2$*, the resulting topology is a bipartite Turán graph.*

In the case of certain topologies, under a given utility model, the conditions required for its formation on discretely small number of nodes, are inconsistent with that required on arbitrarily large number of nodes. Under the proposed utility model, k-star ($k \geq 3$) is one such topology [1]. A possible and reasonable solution to overcome this problem is to analyze the network formation process, starting with a graph that overcomes the conditions required for discretely small number of nodes. This graph can be obtained by some other method, one of which could be providing additional incentives to the nodes of this graph.

Theorem 4 *For a network starting with complete network on* k *centers (*$k \geq 3$*) with the centers connecting to one leaf node each, and* $\gamma = 0$*, if* $c = b_1 - b_3$ *and* $b_2 - b_3 < c_0 < b_2 - b_4$*, the resulting topology is a* k*-star graph.*

The value of c_0 lays the foundation for the degree distribution in a network as it dictates the first connection of a newly entering node. For instance, the values of c_0 in Theorems 1, 3 and 4 ensure that the degree of the first connection of a newly entering node is high, low, and intermediate, respectively. Furthermore, the constraints on γ arise owing to contrasting natures of connectivity in a network. For instance, in a bipartite Turán network, nodes from different partitions are densely connected with each other, while that from the same partition are not connected at all. Similarly, in a k-star network, there is an extreme contrast in the densities of connections (dense amongst centers and sparse for leaf nodes).

4 Discussion and Future work

We proposed a model of recursive network formation where nodes enter a network sequentially, thus triggering evolution of the network each time a new node enters. Though we have assumed a sequential move game model with myopic nodes and pairwise stability as the solution concept, the model, as depicted in Figure 1, is independent of the model of network evolution, the solution concept used for equilibrium state, and also the utility model. The recursive nature of our model enabled us to directly analyze the network formation game using an elegant induction based technique. We derived sufficient conditions for relevant topologies by directing the network evolution along a desired improving path in the sequential move game tree.

It would be interesting to design incentives such that agents in a network comply with the derived sufficient conditions. Our analysis ensures that irrespective of the chosen node at any point in time, the network evolution is directed as desired. A possible solution for simplifying the analysis for more involved topologies is to carry out probabilistic analysis for deriving conditions so that a network has the desired topology with high probability. Another interesting direction, from a practical viewpoint, is to study the problem of forming networks where the topology need not be exactly the one which is ideally desirable, for example, a near-k-star network instead of a precise k-star.

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